ECE 109 Quiz Review #2

(Q1) Does $P(A|B) = P(A|B^c)$ imply A, B independent?
If so, prove it. If not, find a counter example.

A Yes.

Pf: $$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$
$$= P(A|B)P(B) + P(A|B)P(B^c)$$
$$= P(A|B)\left[P(B) + P(B^c)\right]$$
$$= P(A|B)$$

$\square$

Q2 Suppose we roll 2 dice, and the sum is at least 6. What is the probability none of the dies rolled a 1?

A: Use Bayes

Let $A = \{ \text{sum} \geq 6 \}$

$B = \{ \text{no 4's} \}$

$$P(B \mid A) = \frac{P(A \mid B)\, P(B)}{P(A)}$$

$$P(B) = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

$$\frac{10}{36} = \frac{5}{18}$$

Sums: (10 possible)

1 ϕ

2 $\cancel{1+1}$

3 $\cancel{1+2}$ or $\cancel{2+1}$

4 $\cancel{1+3}$, $\cancel{3+1}$, or $2+2$

5 $\cancel{1+4}$, $\cancel{4+1}$, $2+3$, $3+2$

$$P(A) = 1 - P(A^c)$$
$$= 1 - 5/18$$
$$= 13/18$$

$$P(A \mid B) = 1 - P(A^c \mid B)$$
$$= 1 - \frac{3}{25}$$
$$= 22/25$$

So
$$P(B \mid A) = \frac{(22/25)(25/36)}{13/18}$$
$$= \frac{\overset{11}{\cancel{22}} \cdot \cancel{18}}{13 \cdot \cancel{36}\,\cancel{2}}$$
$$= \boxed{11/13}$$

(Q3) Suppose we flipped 5 coins, and 3 of them turned out to be heads. What is the probability that the first flip was a head?

A

Let $A = \{\text{first flip head}\}$
$B = \{3 \text{ heads}\}$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

$P(B|A) \rightsquigarrow$ 2 heads in 4 remaing flips

$$= \frac{\binom{4}{2}}{2^4} = 6/2^4$$

$$= 3/8$$

$$\frac{5 \times \cancel{4}^{2} \times \cancel{3 \times 2}}{\cancel{2} \times \cancel{3 \times 2}}$$

$$P(B) = \frac{\binom{5}{3}}{2^5} = \frac{10}{2^5}$$

$$= 5/16$$

So $$P(A|B) = \frac{\left(\frac{3}{8}\right)\left(\frac{1}{2}\right)}{\frac{5}{16}}$$

$$= \boxed{3/5}$$

Q4 Suppose $S = \mathbb{N}$, and we assciate each $n \in \mathbb{N}$ with probability 2^{-n}. Let E be the event that the outcome is larger than 2. Which of these is independent of E?

a) The event that n is less than 4. ✗

b) The event that n is even ✓

c) The event that n is odd. ✓

A

$$P(E) = \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{4}$$

a) Denote this event as A.

$$P(A) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 7/8$$

$$P(EA) = 1/8$$

$$P(A)P(E) = \frac{7}{8} \cdot \frac{1}{4} \neq \frac{1}{8} \quad \times$$

b) Denote this event as B.

$$P(B) = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$$

$$= \frac{1}{4} \cdot \frac{1}{1 - 1/4}$$

$$= 1/4 \cdot 4/3$$

$$= 1/3$$

$$P(EB) = P(B) - \frac{1}{4}$$

$$= 1/12$$

$$P(E)P(B) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12} = P(EB) \quad ✓$$

c) Denote this event as C.

$$P(C) = 1 - P(B) = 2/3$$

$$P(EC) = P(C) - 1/2 = 1/6$$

$$P(E)P(C) = \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6} = P(EC) \quad ✓$$