ECE 109 Quiz 8 Review

- Topics:
  - Functions of R.V.'s
    - Sums
    - Products
    - Max / min
  - Pairs of functions of R.V.'s
  - Expectation of functions of R.V.'s
  - Covariance & Correlation coefficient

(1) Let X, Y be random variables with joint pmf

$$P_{X,Y}(u,v) = \frac{90\,\pi^{v-4}}{e^{\pi}\,u^4\,v!}$$

$u \in \mathbb{N}$, $v \in \mathbb{N} \cup \{0\}$. Find $\rho_{X,Y}$. (Note: $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$)

Note that we can factor

$$P_{X,Y} = \frac{90}{e^{\pi}\pi^4} \frac{1}{u^4} \frac{\pi^v}{v!} = \underbrace{\left(\frac{90}{\pi^4} \cdot \frac{1}{u^4}\right)}_{\text{valid pdf}} \cdot \underbrace{\left(\frac{\pi^v}{e^{\pi} v!}\right)}_{\text{valid pdf}}$$

And in fact,

$$P_X(u) = \sum_{v=0}^{\infty} \frac{90}{\pi^4} \cdot \frac{1}{u^4} \cdot \frac{\pi^v}{e^{\pi} v!} = \frac{90}{\pi^4} \cdot \frac{1}{u^4} \cdot \frac{1}{e^{\pi}} \cancel{\sum \frac{\pi^v}{v!}}\, e^{\pi} = \frac{90}{\pi^4} \frac{1}{u^4}$$

$$P_Y(v) = \sum_{u=1}^{\infty} \frac{90}{\pi^4} \cdot \frac{1}{u^4} \cdot \frac{\pi^v}{e^{\pi} v!} = \frac{\pi^v}{e^{\pi} v!} \left(\frac{90}{\pi^4} \cdot \frac{\pi^4}{90}\right) = \frac{\pi^v}{e^{\pi} v!}$$

And $P_{X,Y}(u,v) = P_X(u)\,P_Y(v)$ so X, Y ind.

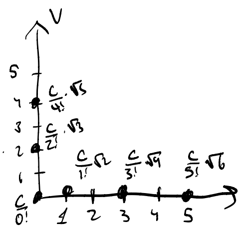
$\Rightarrow \rho_{X,Y} = 0$ ✓

② Let X, Y be random variables with joint pmf

$$P_{X,Y}(u,v) = \begin{cases} \dfrac{C\sqrt{u+v+1}}{(u+v)!} & u=0,\ v \text{ even} \geq 0 \\ & \text{or} \\ & v=0,\ u \text{ odd} \geq 0 \\ 0 & \text{else} \end{cases}$$

For appropriate $C \in \mathbb{R}$. Find some function $f \not\equiv 1$ s.t. $E[f(X,Y)] = 1$.

Look @ pmf



And remember

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

$$E[f(X,Y)] \overset{?}{=} \sum_u \sum_v f(u,v)\, C \cdot \frac{\sqrt{u+v+1}}{(u+v)!}$$

Put $f = \dfrac{1}{Ce\sqrt{u+v+1}}$ for cancellation.

$$\overset{?}{=} \sum_u \sum_v \frac{1}{e\,(u+v)!}$$

$$= \sum_{u=0} \sum_v \frac{1}{e(u+v)!} + \sum_{u>0} \sum_v \frac{1}{e(u+v)!}$$

$$= \sum_{v \text{ even}} \frac{1}{e\,v!} + \sum_{u>0} \frac{1}{e\,u!} \qquad \left(\begin{array}{l} \text{B/c} \quad u>0 \Rightarrow v=0 \\ \qquad\ \ u=0 \Rightarrow v \text{ even} \end{array}\right)$$

$$= \sum_{v \text{ even}} \frac{1}{ev!} + \sum_{u \text{ odd}} \frac{1}{eu!}$$

$$= \frac{1}{e}\left(\frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \cdots\right) + \frac{1}{e}\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right)$$

$$= \frac{1}{e}\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right]$$

$$= \frac{1}{e} \cdot e$$

$$= 1 \quad \checkmark$$

③ let X, Y be independent random variables. let $U = \max(X,Y)$, $V = \min(X,Y)$
Find the joint pdf $f_{U,V}(u,v)$

First find $F_{U,V}$. $P(U \le u, V \le v)$
$= P(\text{Max}(X,Y) \le u, \text{Min}(X,Y) \le v)$

(obviously 0 if $u < v$)

Remember: $\max(X,Y) < a$ is same as $X < a$ and $Y < a$
$\min(X,Y) > a$ is same as $X > a$ and $Y > a$

Can we rewrite in this form?

let $A = \{\text{Max}(X,Y) \le u\}$, $B = \{\text{Min}(X,Y) \le v\}$

$P(A) = P(AB) + P(AB^c)$

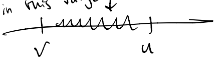
So $P(AB) = P(A) - P(AB^c)$

$\leadsto P(\text{Max} \le u, \text{Min} \le v) = P(\text{Max} \le u) - P(\max \le u, \min > v)$

$= P(\text{Max}(X,Y) \le u) - P(\text{Max}(X,Y) \le u, \min(X,Y) > v)$

$= P(X \le u, Y \le u) - P(v < X \le u, v \le Y \le u)$

Both X,Y have to be in this range



$= P(X \le u)P(Y \le u) - P(v \le X < u)P(v \le Y < u)$

$= F_X(u)F_Y(u) - (F_X(u) - F_X(v))(F_Y(u) - F_Y(v))$

Then $\frac{\partial}{\partial v} F_{U,V}(u,v) = -(-f_X(v))[F_Y(u) - F_Y(v)] - (-f_Y(v))[F_X(u) - F_X(v)]$

$= f_X(v)[F_Y(u) - F_Y(v)] + f_Y(v)[F_X(u) - F_X(v)]$

$\frac{\partial}{\partial u}\frac{\partial}{\partial v} F_{U,V} = f_X(v) f_Y(u) + f_Y(v) f_X(u)$

$\parallel$

$f_{U,V}(u,v)$

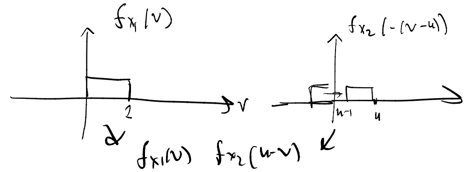
So
$$f_{U,V} = \begin{cases} f_X(v)f_Y(u) + f_Y(v)f_X(u) & u \ge v \\ 0 & \text{else} \end{cases}$$

(4) Let $X_i \sim \text{Unif}[0,1]$ i.i.d. $\forall i = 1 \in \mathbb{N}$, and put $Z_n = \sum_{i=1}^{n} X_i \quad n \in \mathbb{N}$. Find $f_{Z_n}(u)$ for $0 \leq u \leq 1$.

You may use the fact that Z_{n-1} and X_n are independent r.v.

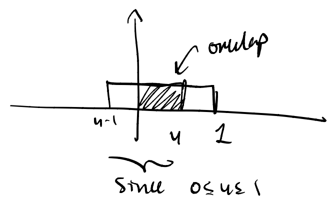
In general: $\quad Z_n = Z_{n-1} + X_n$

Then $\quad f_{Z_n} = f_{Z_{n-1}} * X_n$ convolution



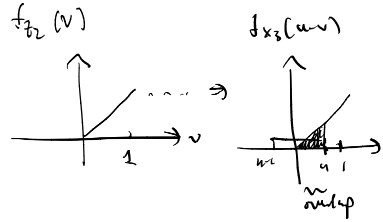
$\underline{\underline{n=2}}$ $(0 \leq u \leq 1)$

$$f_{Z_2}(u) = \int_{-\infty}^{\infty} f_{Z_1}(v)\, f_{X_2}(u-v)\, dv$$
$$= \int_0^u 1 \, du$$
$$= u$$



$\underline{\underline{n=3}}$

$$f_{Z_3}(u) = \int_{-\infty}^{\infty} f_{Z_2}(v)\, f_{X_3}(u-v)\, dv$$
$$= \int_0^u v \cdot 1 \, dv$$
$$= \frac{v^2}{2}\Big|_0^u$$
$$= \frac{u^2}{2}$$



$\underline{\underline{n=4}}$

$$f_{Z_4}(u) = \int_{-\infty}^{\infty} f_{Z_3}(v)\, f_{X_4}(u-v)\, dv$$
$$= \int_0^u \frac{v^2}{2} \cdot 1 \, dv$$
$$= \frac{1}{2} \cdot \frac{1}{3}\, v^3 \Big|_0^u$$
$$= \frac{1}{3 \cdot 2} u^3$$

In general, guess: $\quad f_{Z_n}(u) = \frac{1}{(n-1)!} u^{n-1}$

by induction: $\quad f_{Z_n}(u) = \int_{-\infty}^{\infty} f_{Z_{n-1}}(v)\, f_{X_n}(u-v)\, dv$

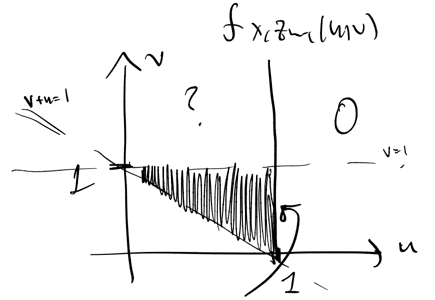
$$= \int_0^u \frac{1}{(n-2)!} v^{n-2} \cdot 1 \, dv$$
$$= \frac{1}{(n-2)!} \cdot \frac{1}{n-1} \cdot v^{n-1} \Big|_0^u$$
$$= \frac{1}{(n-1)!} u^{n-1}$$

✓

⑤ Consider the experiment where we repeatedly sample the interval $[0,1]$ and compute the running sum. Let Y be the number of draws until the running sum first exceeds 1. What is $E[Y]$? (Hint: Use answer from ④)

Let $X_i \sim \text{unif}([0,1])$ $\forall i \in \mathbb{N}$ and put $Z_n = \sum_{i=1}^{n} X_i$ again.

Find pmf of Y.



Shaded region to compute probability over

$$P(Y=n) = P(Z_{n-1} \le 1,\ Z_n > 1)$$
$$= P(Z_{n-1} \le 1,\ X + Z_{n-1} > 1)$$
$$= P(Z_{n-1} \le 1,\ Z_{n-1} > 1 - X)$$
$$= \int_0^1 \int_{1-v}^1 1 \cdot \frac{v^{n-2}}{(n-2)!}\, du\, dv$$
$$= \frac{1}{(n-2)!} \int_0^1 v^{n-1}\, dv$$
$$= \frac{1}{(n-2)!} \cdot \frac{1}{n} \cdot (1-0)$$
$$= \frac{1}{n(n-2)!}$$

Obviously $P(Y \le 1) = 0$

$$\Rightarrow E[Y] = \sum_{n=2}^{\infty} n \cdot P(Y=n)$$
$$= \sum_{n=2}^{\infty} \frac{\cancel{n}}{\cancel{n}(n-2)!}$$
$$= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \qquad k = n-2$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!}$$
$$= e$$

✓