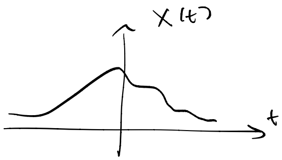
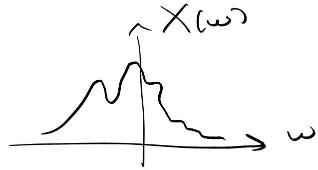
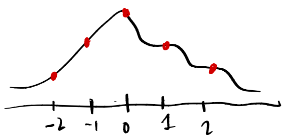
Special Topic: Poisson Summation Formula





Idea: $x(t)$ is not periodic, but what if I constructed a periodic function out of it for fun?

Consider $\displaystyle p(t) = \sum_{k=-\infty}^{\infty} x(t-k)$



$p(0) = x(0) + x(1) + x(-1) + x(2) + x(-2) + \cdots$

What about $p(1)$?

$p(1) = x(1) + x(0) + x(2) + x(-1) + x(3) + \cdots$

This is just a rearrangement of $p(0)$!

In general, $\underline{p(t+1) = p(t)}$

So p is periodic (with period 1) and we have constructed a new periodic function out of $x(t)$

↝ What does its fourier series look like?

* Before proceeding, lets make the technical assumption that $x(t)$ and $X(\omega)$ "decay nicely" so that things don't blow up to infinity. Moreo

More precisely, $\exists\, c, \varepsilon > 0$ s.t. $|x(t)| < \dfrac{c}{(1+|t|)^{1+\varepsilon}}$, $|X(\omega)| < \dfrac{c}{(1+|\omega|)^{1+\varepsilon}}$ $\forall t, \omega \in \mathbb{R}$

non-example: $\cos(2\pi t)$ does not decay nicely

$\displaystyle p(0) = \sum_{k=-\infty}^{\infty} \cos(2\pi k) = \sum_{k=-\infty}^{\infty} 1 = \infty$

Now let's go back to finding the fourier coefficients of P

Here, $\omega_0 = \frac{2\pi}{T} = 2\pi$

So

$$P_n = \frac{1}{1}\int_0^1 P(t)\, e^{-2\pi jnt}\, dt$$

$$= \int_0^1 \sum_{k=-\infty}^{\infty} x(t-k)\, e^{-2\pi jnt}\, dt$$

$$= \sum_{k=-\infty}^{\infty} \int_0^1 x(t-k)\, e^{-2\pi jnt}\, dt$$

Letting $\tau = t-k$

$t=0 \rightarrow \tau = -k$

$t=1 \rightarrow \tau = 1-k$

$$= \sum_{k=-\infty}^{\infty} \int_{-k}^{1-k} x(\tau)\, e^{-2\pi jn(\tau+k)}\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} \int_{-k}^{1-k} x(\tau)\, e^{-2\pi jn\tau}\, \cancel{e^{-2\pi jnk}}\, d\tau$$

this is 1 since n, k are integers!

$$= \sum_{k=-\infty}^{\infty} \int_{-k}^{1-k} x(\tau)\, e^{-2\pi jn\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-2\pi jn\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\underbrace{(2\pi n)}_{\omega}\tau}\, d\tau$$

$$= X(\underbrace{2\pi n})$$

Now, writing P as a fourier series:

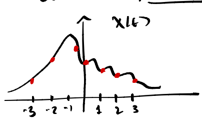
$$P(t) = \sum_{n=-\infty}^{\infty} P_n\, e^{j\omega_0 nt}$$

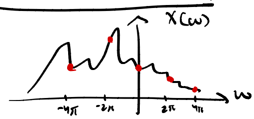
$$= \sum_{n=-\infty}^{\infty} X(2\pi n)\, e^{2\pi jnt}$$

$\parallel$ also $\sum_{n=-\infty}^{\infty} x(t-n)$ (by definition)

So $$\boxed{\sum_{n=-\infty}^{\infty} x(t-n) = \sum_{n=-\infty}^{\infty} X(2\pi n)\, e^{2\pi jnt}}$$ ← Poisson summation formula

$\underline{t=0}$: $$\boxed{\sum_{n=-\infty}^{\infty} x(n) = \sum_{n=-\infty}^{\infty} X(2\pi n)}$$

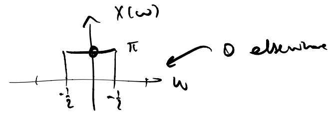




These sums are the same!

$\underline{Ex}$ $\quad x(t) = \text{sinc}(t), \quad X(\omega) = \pi\, \text{rect}(\omega/2)$

Then $$\sum_{n=-\infty}^{\infty} \text{sinc}(n) = \pi$$



So $$\sum_{n=-\infty}^{-1} \frac{\sin(n)}{n} + 1 + \sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \pi$$

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \frac{1}{2}(\pi - 1)}$$

:)

$\underline{\text{Note}}$: The Sampling Theorem, which we'll see week 7, can be viewed as a reformulation of the Poisson Summation formula. However, we'll derive it a different way in lecture.