Lecture 17 Review

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

ROC for $\mathcal{L}\{x(t)\}$ contains $Re(s) = 0$
$\Rightarrow \mathcal{F}\{x(t)\}$ exist

$$\mathcal{L}\{x(t)\}\Big|_{s=j\omega} = \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt}_{\mathcal{F}\{x(t)\}}$$

$$\boxed{\begin{array}{c} e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \\ \text{ROC: } Re(s) > Re(a) \end{array}}$$ causal case

$$\boxed{\begin{array}{c} -e^{-at} u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \\ \text{ROC: } Re(s) < Re(a) \end{array}}$$ anti-causal case

Pf 2nd part:

$$\mathcal{L}\{-e^{-at} u(-t)\} = \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-at} e^{-st}\, dt$$

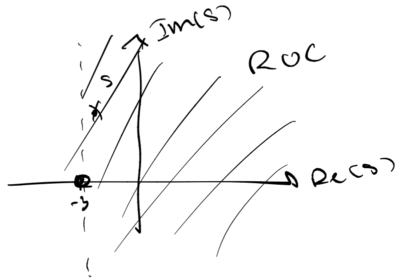
$$= -\int_{-\infty}^{0} e^{-t(a+s)}\, dt$$

$$= \frac{1}{a+s}\left[e^{0} - \cancel{e^{\infty(a+s)}}^{\,0}\right]$$

$$= \frac{1}{a+s}$$ provided $Re(a+s) < 0$

$Re(s) < Re(-a)$

Ex What is $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$ given that the transform exists at $s = -e + \pi j$?



So the ROC must be $Re(s) > -3$

So it's the causal case

So $\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$

$= e^{-3t} u(t)$

$$\boxed{u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \text{ROC: } Re(s) > 0}$$

$$\boxed{-u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \text{ROC: } Re(s) < 0}$$

Pf Plug in $a = 0$ in previous pairs.

$$\boxed{\sin(at)u(t) \xleftrightarrow{\mathcal{L}} \frac{a}{s^2+a^2} \quad Re(s) > 0} \quad a \in \mathbb{R}$$

Pf using Euler's formula + linearity [Lec 18]

$$\sin(at) = \frac{1}{2j}\left(e^{ajt} - e^{-ajt}\right)$$

$$\sin(at)u(t) = \frac{1}{2j} e^{ajt} u(t) - \frac{1}{2j} e^{-ajt} u(t)$$

$$\mathcal{L}\{\sin(at)u(t)\} = \frac{1}{2j}\left[\frac{1}{s-aj}\right] - \frac{1}{2j}\left[\frac{1}{s+aj}\right]$$

$$= \frac{1}{2j}\left[\frac{s+aj-(s-aj)}{(s-aj)(s+aj)}\right] = \frac{1}{2j}\frac{2aj}{s^2+a^2} = \frac{a}{s^2+a^2} \quad \blacksquare$$

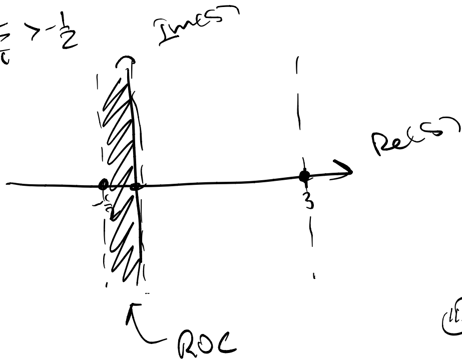
Similarly: $\boxed{\cos(at)\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+a^2}, \quad \text{ROC } \mathrm{Re}(s) > 0}$

PS ★ Ex

Ex Suppose $X(s) = \frac{2}{1+2s} + \frac{4}{3-s} + \frac{j}{s}$ and $-e\,(3j+1)/2\pi$ lies in the ROC. Find the inverse $x(t)$.

$\frac{e}{2\pi} < \frac{1}{2}$

$-\frac{e}{2\pi} > -\frac{1}{2}$



i) $\frac{2}{1+2s}$: Causal case

$= \frac{1}{\frac{1}{2}+s} \longleftrightarrow e^{-\frac{1}{2}t}u(t)$

ii) $\frac{4}{3-s}$: anti-causal case

$= \frac{-4}{s-3} \longleftrightarrow 4e^{3t}u(-t)$

iii) $\frac{j}{s}$ anti-causal

$\longleftrightarrow -j\,u(-t)$

So $x(t) = e^{-\frac{1}{2}t}u(t) + 4e^{3t}u(-t) - j\,u(-t)$

Ex If $X(s) = \frac{s+1}{s+2} - \frac{s}{s+3}$, what is its anti-causal inverse?

$X(s) = \frac{s+2-1}{s+2} - \frac{s+3-3}{s+3}$

$= 1 - \frac{1}{s+2} - 1 + \frac{3}{s+3}$

$= -\frac{1}{s+2} + \frac{3}{s+3}$

$= -(-e^{-2t}u(-t)) + 3(-e^{-3t}u(-t))$

$= (e^{-2t} - 3e^{-3t})\,u(-t)$

anti-causal pair:

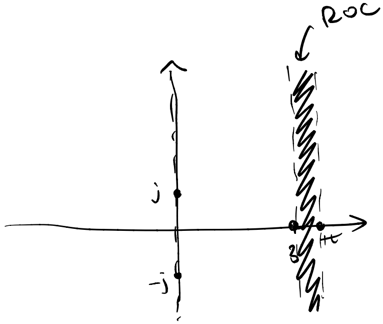
$-e^{-at}u(-t) \longleftrightarrow \frac{1}{s+a}$

Ex If $x(t) = 4e^{3t+2}u(t) - \pi^e e^{\pi t} u(-t) - \cos(-t)u(t)$, find its region of convergence.

$4e^{3t+2}u(t)$

$= (4e^2)e^{3t}u(t) \longleftrightarrow \dfrac{4e^2}{s-3}$

ROC: $Re(s) > 3$



(in general $\mathcal{L}\{e^{at}u(\pm t)\}$ has a pole at $s=a$)

why? $\mathcal{L}\{e^{at}u(t)\} = \frac{1}{s-a}$ pole @ $s=a$

$\mathcal{L}\{e^{at}u(-t)\} = -\frac{1}{s-a}$ pole @ $s=a$

$\mathcal{L}\{\pi^e e^{\pi t}u(-t)\}$ has a pole at $s=\pi$

anti-causal case $\Rightarrow$ ROC: $Re(s) < \pi$

$\mathcal{L}\{\cos(-t)u(t)\} = \mathcal{L}\{\cos(t)u(t)\}$ has poles at $\pm j$

ROC: $Re(s) > 0$

ROC: $3 < Re(s) < \pi$

$\cos(t)u(t) = \frac{1}{2}(e^{jt} + e^{-jt})u(t)$

$= \frac{1}{2}e^{jt}u(t) + \frac{1}{2}e^{-jt}u(t)$

$\mathcal{L}\{\frac{1}{2}e^{jt}u(t)\} = \frac{1}{2}\cdot\frac{1}{s-j}$ ROC: $Re(s) > Re(j) = 0$

$\mathcal{L}\{\frac{1}{2}e^{-jt}u(t)\} = \frac{1}{2}\frac{1}{s+j}$ ROC: $Re(s) > Re(-j) = 0$

$\Big\}$ $Re(s) > 0$

$$\mathcal{L}\{\sin(at)u(-t)\}$$

$$\frac{1}{2j}(e^{ajt} - e^{-ajt})u(-t)$$

$$\mathcal{L}\left\{\frac{1}{2j}e^{ajt}u(-t)\right\} = \frac{1}{2j}\,\frac{-1}{s-aj} \qquad \text{ROC: } Re(s) < \cancel{Re(aj)}\ 0$$

$$\mathcal{L}\left\{-\frac{1}{2j}e^{-ajt}u(-t)\right\} = \frac{1}{2j}\,\frac{1}{s+aj} \qquad \underline{\text{ROC}}\text{: } Re(s) < 0$$

So $\mathcal{L}\{\sin(at)u(-t)\} = \frac{1}{2j}\left[\frac{1}{s+aj} - \frac{1}{s-aj}\right]$

$$= \frac{1}{2j}\,\frac{\cancel{s}-aj - (\cancel{s}+aj)}{s^2+a^2}$$

$$= -\frac{a}{s^2+a^2} \qquad \underline{\text{ROC}}\text{: } Re(s) < 0$$

$$\boxed{-\sin(at)\,u(-t) \longleftrightarrow \frac{a}{s^2+a^2} \qquad \text{ROC: } Re(s) < 0}$$