ECE 45 Quiz Review 1

Topics:

- Complex numbers review
- Frequency response of an RLC circuit

1) Perform the following operations:

a) $e^{\frac{\pi}{6}j} + e^{-\frac{\pi}{6}j}$

b) $\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

c) $e^{\frac{\pi}{6}j} + e^{\frac{\pi}{3}j} + e^{\frac{2\pi}{3}j} + e^{\frac{5\pi}{6}j}$

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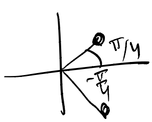
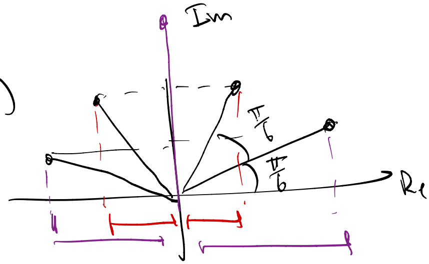
a) Recall: $\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

$$e^{\frac{\pi}{6}j} + e^{-\frac{\pi}{6}j} = 2\cos\left(\frac{\pi}{6}\right) = 2 \cdot \sqrt{3}/2 = \sqrt{3}$$

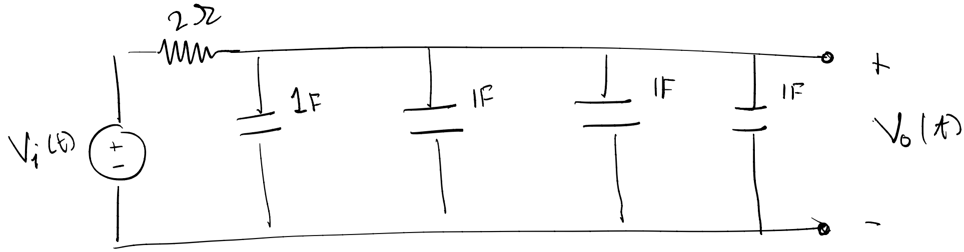
b) 

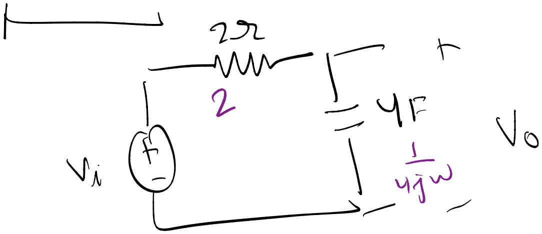
$$e^{\frac{\pi}{4}j} \cdot e^{-\frac{\pi}{4}j} = 1$$

c) 

$$\left(2\sin\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{3}\right)\right)j = (1 + \sqrt{3})j$$

(2) Find the frequency response for the following RLC circuit.





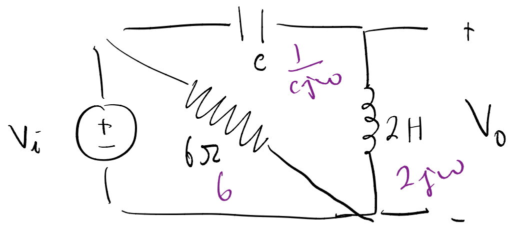
equivalent circuit

voltage divider: $$V_o = \frac{\frac{1}{4j\omega}}{2 + \frac{1}{4j\omega}} V_i$$

$$= \boxed{\frac{1}{8j\omega + 1}} V_i$$

$$= H(\omega)$$

③ Find the smallest capacitance s.t. the magnitude of the circuit's frequency response is 4 when $V_i(t) = \sqrt{\pi}^{e} \sin(t - \pi/4)$



$$V_o = \frac{2j\omega}{\frac{1}{Cj\omega} + 2j\omega} V_i$$

$$H(\omega) = \frac{2j\omega}{\frac{1}{Cj\omega} + 2j\omega}$$

$$= \frac{-2C\omega^2}{1 - 2C\omega^2}$$

$$H(1) = \frac{-2C}{1-2C} = \frac{2C}{2C-1}$$

Want $|H(1)| = 4$

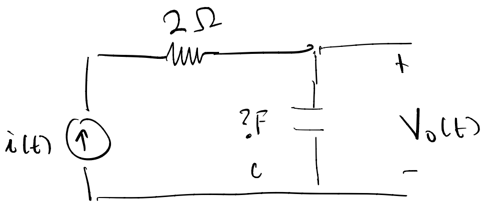
$$\left|\frac{2C}{2C-1}\right| = 4$$

$$\frac{2C}{2C-1} = 4 \quad \text{or} \quad \frac{2C}{2C-1} = -4$$

$$2C = 8C - 4 \qquad 2C = -8C + 4$$

$$6C = 4 \rightarrow C = 2/3 \quad \text{or} \quad 10C = 4 \rightarrow \boxed{C = \frac{2}{5}}$$

(4) In the following circuit, an input of $i(t) = 2\cos(4t+\pi)$ gives an output voltage of $V_o(t) = \sqrt{3}\cos(4t+\frac{\pi}{2})$. Find $H(\omega)$.



$$V_o = \left(\frac{1}{Cj\omega}\right) I$$

$$H(\omega) = \frac{1}{Cj\omega}$$

$$H(4) = \frac{1}{4Cj}$$

But we know $H(4)$

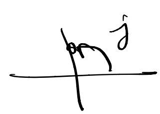
$$2\cos(4t+\pi) \longleftrightarrow 2e^{\pi j}$$

$$\sqrt{3}\cos(4t+\frac{\pi}{2}) \longleftrightarrow \sqrt{3}e^{\frac{\pi}{2}j}$$

$$2e^{\pi j} \cdot H(4) = \sqrt{3}e^{+\frac{\pi}{2}j}$$

$$H(4) = \frac{\sqrt{3}}{2}e^{-\frac{\pi}{2}j}$$

$$= \frac{\sqrt{3}}{2j}$$



$$H(4) = \frac{1}{4Cj} = \frac{\sqrt{3}}{2j} \Rightarrow 4C = \frac{2}{\sqrt{3}} \Rightarrow C = \frac{1}{2\sqrt{3}} \Rightarrow H(\omega) = \frac{2\sqrt{3}}{j\omega}$$

(5) (BONUS) Prove the triangle inequality and the reverse triangle inequality for complex numbers - i.e., given $x, y \in \mathbb{C}$, we have $|x-y| \leq |x|+|y|$ and $||x|-|y|| \leq |x-y|$.

Pf $|x-y|^2 = (x-y)(x-y)^*$ (using $|z|^2 = z \cdot z^*$)

$= (x-y)(x^*-y^*)$

$= (xx^* - yx^* - xy^* - yy^*)$

$= |x|^2 - x^*y - xy^* + |y|^2$

$= |x|^2 - (x^*y + (x^*y)^*) + |y|^2$

$= |x|^2 - 2\,\mathrm{Re}(x^*y) + |y|^2$

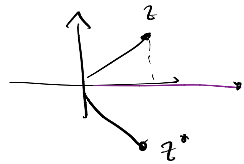
$\leq |x|^2 + 2|x^*y| + |y|^2$

$= |x|^2 + 2|x^*||y| + |y|^2$

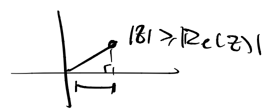
$= |x|^2 + 2|x||y| + |y|^2$

$= (|x|+|y|)^2$

Note: $z + z^* = 2\,\mathrm{Re}(z)$



$|x-y| \leq |x|+|y|$ ✓



Now note that

$|x| = |(x-y)+y| \leq |x-y| + |y|$

$|x| - |y| \leq |x-y|$

$|y| = |(y-x)+x| \leq |y-x| + |x|$

$|y| - |x| \leq |x-y|$

$-|x-y| \leq |x| - |y| \leq |x-y|$

So $||x|-|y|| \leq |x-y|$

$\square$