ECE 45 Quiz Review #5

Topics:
- Fourier transform properties
  - DC value in time/frequency
  - Duality
  - time/frequency derivatives
  - conjugation
  - FT of real signal / real + even signal / imaginary + odd signal
  - time scaling
- Common Fourier Transforms
  - sin / cos / $\delta(t)$
  - rect / sinc
  - $e^{-at}u(t)$
- More on convolutions
  - FT of products / convolutions

High level overview of what we have done so far!

goal: given $x(t)$ as the input to an LTI system, how do I find the output $y(t)$?

cases

① eigenfunction

$$e^{j\omega t} \longrightarrow H(\omega)e^{j\omega t}$$

② $x(t)$ periodic — this is just linearity + eigenfunction property!

$$x(t) = \sum_n x_n e^{jn\omega_0 t} \longrightarrow \sum_n H(n\omega_0)e^{jn\omega_0 t}$$

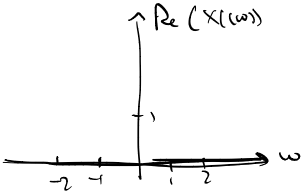
③ $x(t)$ not periodic — eigenfunction property again!

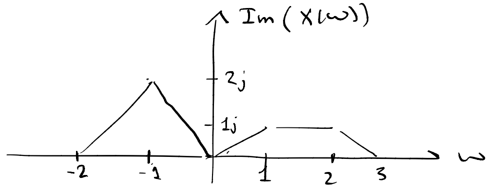
$$x(t) = \frac{1}{2\pi}\int X(\omega)e^{j\omega t}d\omega \longrightarrow \underbrace{\frac{1}{2\pi}\int H(\omega)X(\omega)e^{j\omega t}d\omega}_{=\mathcal{F}^{-1}\{H(\omega)X(\omega)\} = x(t) * h(t)}$$

giving us the following "commutative diagram"

$$\begin{tikzcd}
\text{time domain} & x(t) \arrow[r] \arrow[d, shift right, "\mathcal{F}"'] & \boxed{h(t)} \arrow[r] \arrow[d, shift right, "\mathcal{F}"'] & y(t) = x(t)*h(t) \arrow[d, shift right, "\mathcal{F}"'] \\
\text{frequency domain} & X(\omega) \arrow[r] \arrow[u, shift right, "\mathcal{F}^{-1}"'] & \boxed{H(\omega)} \arrow[r] \arrow[u, shift right, "\mathcal{F}^{-1}"'] & Y(\omega) = X(\omega)H(\omega) \arrow[u, shift right, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

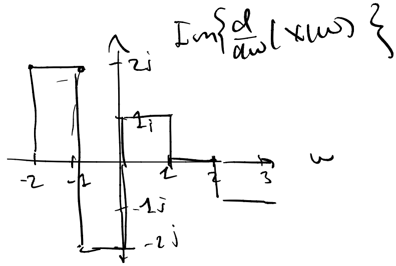
1. Below is the graph for $X(\omega) := \mathcal{F}\{x(t)\}$. If $H(\omega) = \omega\,\mathrm{rect}(\omega/2)$, find the output of the system @ time 0 when the input is $t\,x(t)$.

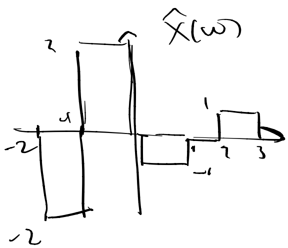


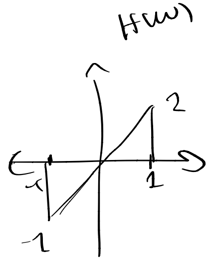


A. Let $\hat{x}(t) = t\,x(t)$

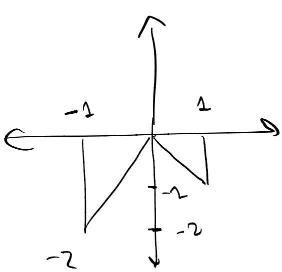
then $\hat{X}(\omega) = j \frac{d}{d\omega} X(\omega)$







$\hat{y}(\omega) = \hat{X}(\omega)\,H(\omega)$



$$\hat{y}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(\omega) e^{0j\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int \hat{y}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\left[-1 - \frac{1}{2}\right]$$

$$= \boxed{-\frac{3}{4\pi}}$$

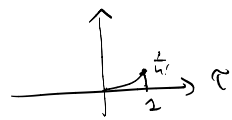
② Find $\frac{1}{n!} t^n \operatorname{rect}(t-\frac{1}{2}) * [u(t) - u(t-1)]$

A. Denote $x(t) = u(t) - u(t-1)$

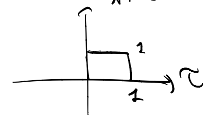
$$y(t) = \frac{1}{n!} t^n \operatorname{rect}(t-\tfrac{1}{2}) * x(t)$$

$$= \int_{-\infty}^{\infty} \frac{1}{n!} \tau^n \operatorname{rect}(\tau-\tfrac{1}{2})\, x(t-\tau)\, d\tau$$

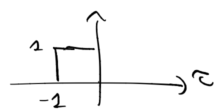
$\frac{1}{n!} \tau^n \operatorname{rect}(\tau-\frac{1}{2})$



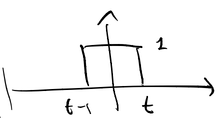
$x(\tau)$



$x(-\tau)$

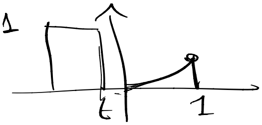


$x(t-\tau)$



case 1

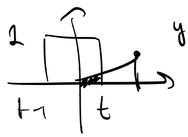
$t < 0$



$y(t) = 0$

case 2

$t > 0$
$t < 1$

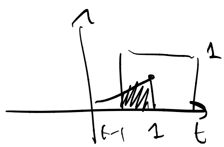


$$y(t) = \int_0^t \frac{1}{n!} \tau^n \, d\tau$$

$$= \frac{1}{(n+1)!} t^{n+1}$$

Case 3

$t > 1$
$t-1 < 1$

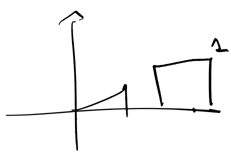


$$y(t) = \int_{t-1}^{1} \frac{1}{n!} \tau^n \, d\tau$$

$$= \frac{1}{(n+1)!}\left[1 - (t-1)^{n+1}\right]$$

Case 4

$t-1 > 1$



$y(t) = 0$

$$\boxed{y(t) = \begin{cases} 0 & t < 0 \text{ or } t > 2 \\ \frac{1}{(n+1)!} t^{n+1} & 0 < t < 1 \\ \frac{1}{(n+1)!}\left(1 - (t-1)^{n+1}\right) & 1 < t < 2 \end{cases}}$$

③ Suppose I have an LTI system in which the input $x(t)$ and the output $y(t)$ are governed by the following differential equation:

$$y''(t) = 3y'(t) - y(t) + x(t-1)$$

Find the output of the system when $x(t) = A\cos(\omega_0 t)$.

A. Applying the Fourier Transform both sides, we have

$$(j\omega)^2 Y(\omega) = 3j\omega Y(\omega) - Y(\omega) + e^{-j\omega} X(\omega)$$

$$Y(\omega)\left[-\omega^2 - 3j\omega + 1\right] = e^{-j\omega} X(\omega)$$

$$Y(\omega) = \underbrace{\frac{e^{-j\omega}}{-\omega^2 - 3j\omega + 1}}_{H(\omega)} X(\omega)$$

So if $x(t) = A\cos(\omega_0 t)$, then

$$X(\omega) = A\pi\,\delta(\omega-\omega_0) + A\pi\,\delta(\omega+\omega_0)$$

$$Y(\omega) = \frac{A\pi\,\delta(\omega-\omega_0)e^{-j\omega_0}}{-\omega_0^2 - 3j\omega_0 + 1} + \frac{A\pi\,\delta(\omega+\omega_0)\,e^{j\omega_0}}{-\omega_0^2 + 3j\omega_0 + 1}$$

Then $$y(t) = \left[\frac{A\,e^{-j\omega_0}}{2(-\omega_0^2 - 3j\omega_0 + 1)}\right] e^{j\omega_0 t} + \left[\frac{e^{j\omega_0}}{2(-\omega_0^2 + 3j\omega_0 + 1)}\right] e^{-j\omega_0 t}$$

Alternatively:

$$x(t) = \frac{A}{2}e^{j\omega_0 t} + \frac{A}{2}e^{-j\omega_0 t}$$

$$\downarrow$$

$$y(t) = H(\omega_0)\cdot\frac{A}{2}e^{j\omega_0 t} + H(-\omega_0)\,\frac{A}{2}e^{-j\omega_0 t}$$

(eigenfunction property!)

(4) Compute $\mathcal{F}^{-1}\left\{\frac{-1}{w^2-5jw-6}\right\}$

A. $$w^2-5jw-6 = -(jw)^2-5jw-6 \leftarrow \text{(Polynomial in } jw)$$
$$= -\left((jw)^2+5jw+6\right)$$
$$= -(jw+3)(jw+2)$$

Write $$-\frac{1}{w^2-5jw-6} = \frac{1}{(jw+3)(jw+2)} = \frac{A}{jw+3}+\frac{B}{jw+2}$$

$$1 = A(jw+2)+B(jw+3) \quad \forall w$$

Re: $1 = 2A+3B$

Im: $0 = Aw+Bw$ (holds for all w, so must hold for $w=1$)

$A+B=0$

$A=-B$

$1 = -2\cancel{B}+3\cancel{B}$

$\underline{\underline{B=1}}$ $\quad$ $\underline{\underline{A=-1}}$

$$\mathcal{F}^{-1}\left\{-\frac{1}{w^2-5jw-6}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{jw+2}-\frac{1}{jw+3}\right\}$$
$$= e^{-2t}u(t) - e^{-3t}u(t)$$
$$= \left[e^{-2t}-e^{-3t}\right]u(t)$$

⑤ Suppose $x(t) = \sum_{n=0}^{\infty} 2^{-n} \cos(t - \frac{\pi}{2} n)$ is the input to a system with frequency response $H(\omega) = \cos(\frac{\pi}{2}\omega)$. Find the output $y(t)$.

A.
$$X(\omega) = \sum_{n=0}^{\infty} 2^{-n} \pi [\delta(\omega-1) + \delta(\omega+1)] e^{-jn\frac{\pi}{2}\omega}$$
$$= \pi [\delta(\omega-1) + \delta(\omega+1)] \left(\sum_{n=0}^{\infty} \left(\frac{1}{2e^{\frac{\pi}{2}j\omega}} \right)^n \right)$$
$$= \pi [\delta(\omega-1) + \delta(\omega+1)] \frac{1}{1 - \frac{1}{2e^{\frac{\pi}{2}j\omega}}}$$
$$= \pi [\delta(\omega-1) + \delta(\omega+1)] \frac{2e^{\frac{\pi}{2}j\omega}}{2e^{\frac{\pi}{2}j\omega} - 1}$$
$$= \pi \frac{2e^{\frac{\pi}{2}j}}{2e^{\frac{\pi}{2}j} - 1} \delta(\omega-1) + \pi \cdot \frac{2e^{-\frac{\pi}{2}j}}{2e^{-\frac{\pi}{2}j} - 1} \delta(\omega+1)$$
$$= \pi \cdot \frac{2j}{2j-1} \delta(\omega-1) - \pi \cdot \frac{2j}{2j+1} \delta(\omega+1)$$

$$Y(\omega) = H(\omega) X(\omega)$$
$$= \cos(\frac{\pi}{2}\omega) \left[\frac{2\pi j}{2j-1} \delta(\omega-1) - \frac{2\pi j}{2j+1} \delta(\omega+1) \right]$$
$$= \cos(\frac{\pi}{2}) \cdot \frac{2\pi j}{2j-1} \delta(\omega-1) - \cos(-\frac{\pi}{2}) \frac{2\pi j}{2j+1} \delta(\omega+1)$$
$$= 0$$

So $y(t) = 0$