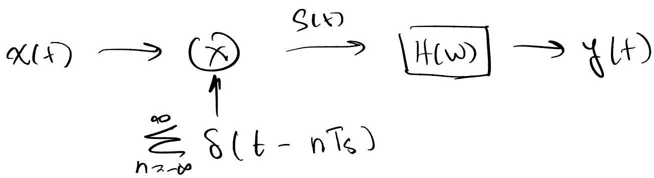
ECE 45 Quiz Review 8

① Say I sample a signal $x(t)$ (with F.T. $X(\omega) = \text{rect}(\frac{\omega}{20})$) w/ $T_s = \frac{2\pi}{19}$. If I try to recover my signal using a LPF with $H(\omega) = T_s \, \text{rect}(\omega/20)$, then what is my output signal $y(t)$? What is the error $|y(t) - x(t)|$ due to aliasing?

A.

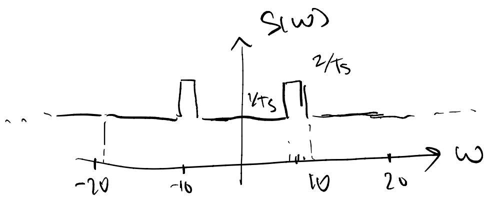


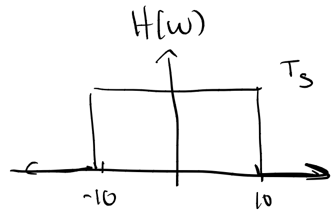
$$s(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

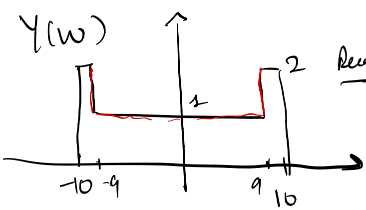
$$S(\omega) = \frac{1}{2\pi}\left[X(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(t - n\omega_s)\right] \quad \text{where} \quad \omega_s = \frac{2\pi}{T_s} = 19$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - 19s)$$

$$Y(\omega) = H(\omega) S(\omega)$$







$$Y(\omega) = 2\,\text{rect}\left(\frac{\omega}{20}\right) - \text{rect}\left(\frac{\omega}{18}\right)$$

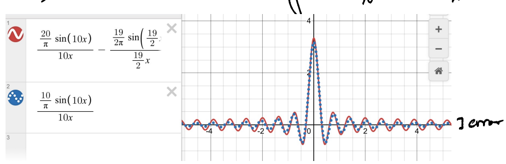
Recall: $\frac{a}{\pi} \text{sinc}(at) \longleftrightarrow \text{rect}\left(\frac{\omega}{2a}\right)$

So $2 \cdot \frac{10}{\pi} \text{sinc}(10t) \longleftrightarrow 2\,\text{rect}(\omega/20)$

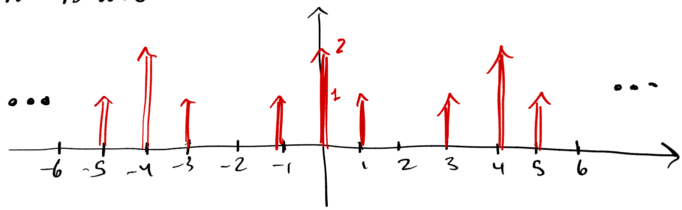
$-\frac{9}{\pi} \text{sinc}(9t) \longleftrightarrow -\text{rect}(\omega/18)$

So $y(t) = \frac{20}{\pi} \text{sinc}(10t) - \frac{9}{\pi} \text{sinc}(9t)$

Since $X(\omega) = \text{rect}(\frac{\omega}{20})$, $x(t) = \frac{10}{\pi} \text{sinc}(10t) \rightarrow$ error due to aliasing: $\left|\frac{10\,\text{sinc}(10t)}{\pi} - \frac{9}{\pi}\text{sinc}(9t)\right|$

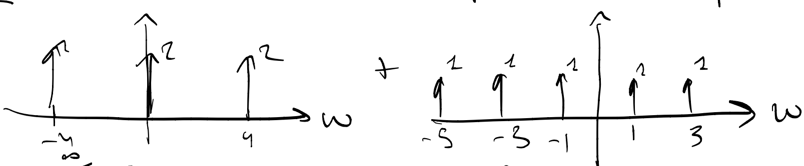


② Suppose $F(\omega)$ is periodic with period 4, as shown below.



If $\mathcal{F}\{f(t)\} = F(\omega)$, then what is $f(1)$?

A: We can view $F(\omega)$ as a sum of two impulse.



$$2\sum_{n=-\infty}^{\infty} \delta(\omega-4n) \quad + \quad \sum_{n=-\infty}^{\infty} \delta(\omega-2n-1) \quad = \quad F(\omega)$$

Remember that

$$\sum_{n=-\infty}^{\infty} \delta(t-nT_s) \longleftrightarrow \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \qquad \omega_s = \frac{2\pi}{T_s}$$

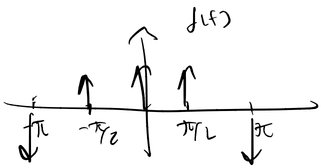
If $\omega_s = 4$, $T_s = \frac{2\pi}{4} = \frac{\pi}{2}$

If $\omega_s = 2$, $T_s = \frac{2\pi}{2} = \pi$

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} \delta\left(t-\frac{\pi}{2}n\right) \longleftrightarrow 2\sum_{n=-\infty}^{\infty} \delta(\omega-4n)$$

$$\frac{1}{2}e^{jt}\sum_{n=-\infty}^{\infty} \delta(t-\pi n) \longleftrightarrow \sum_{n=-\infty}^{\infty} \delta(\omega-2n-1)$$

So $f(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta\left(t-\frac{\pi}{2}n\right) + \frac{1}{2}e^{jt}\sum_{n=-\infty}^{\infty} \delta(t-\pi n)$



No δ's centered @ 1

So $\underline{f(1) = 0}$

③ Compute $\int_{-3}^{3} \mathcal{F}\{\cos^6(t)\}\, d\omega$

A Recall that $\cos(t) = \frac{1}{2}(e^{jt} + e^{-jt})$

first let $x = e^{jt}$: $\hookrightarrow = \frac{1}{2}(x + \frac{1}{x})$

So $\cos^6(t) = \frac{1}{2}(x + \frac{1}{x})^6$

$$= \frac{1}{2^6}\Big[x^6 + 6x^5(\tfrac{1}{x})^1 + \binom{6}{2}x^4(\tfrac{1}{x})^2 + \binom{6}{3}x^3(\tfrac{1}{x})^3 + \binom{6}{4}x^2(\tfrac{1}{x})^4 + 6x(\tfrac{1}{x})^5 + (\tfrac{1}{x})^6\Big]$$

$$= \frac{1}{2^6}\big[x^6 + 6x^4 + 15x^2 + 20 + 15x^{-2} + 6x^{-4} + x^{-6}\big]$$

$$x = e^{jt} \longleftrightarrow 2\pi\,\delta(\omega - 1)$$

$$\forall n \in \mathbb{Z} : \quad x^n = e^{jnt} \longleftrightarrow 2\pi\,\delta(\omega - n)$$

So $\mathcal{F}\{\cos^6(t)\} = \frac{2\pi}{2^6}\big[\delta(\omega-6) + 6\delta(\omega-4) + 15\delta(\omega-2) + 20\delta(\omega) + 15\delta(\omega+2) + 6\delta(\omega+4) + \delta(\omega+6)\big]$

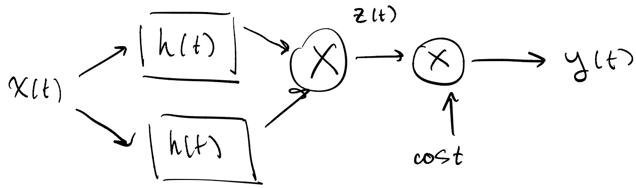
$$\int_{-3}^{-3} \mathcal{F}\{\cos^6(t)\}\, d\omega = \frac{\pi}{2^5}\int_{-3}^{3} 15\delta(\omega-2) + 20\delta(\omega) + 15\delta(\omega+2)\, d\omega$$

$$= \frac{\pi}{2^5}(15 + 20 + 15)$$

$$= \frac{\pi}{2^5}(50)$$

$$= \boxed{25\pi/16}$$

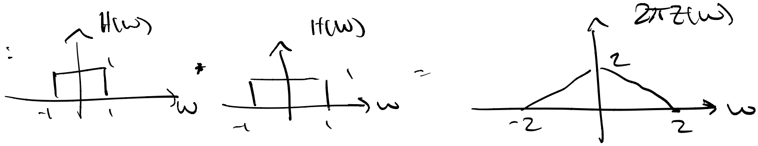
(4) If $x(t) = \delta(t)$ and $h(t)$ has fourier transform $\text{rect}(\omega/2)$ then what is $Y(\omega)$ below?



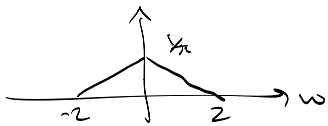
A. $z(t) = h(t) \cdot h(t)$

$Z(\omega) = \frac{1}{2\pi} H(\omega) * H(\omega)$

Recall:



So $Z(\omega)$ looks like



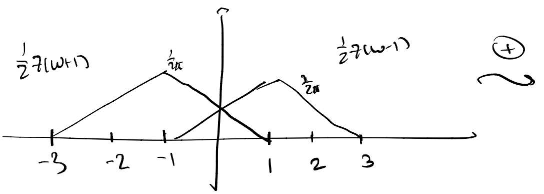
has area $4 \cdot \frac{1}{\pi} \cdot \frac{1}{2}$

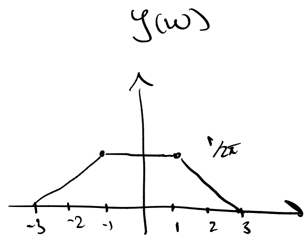
$= 2/\pi$

$y(t) = z(t) \cos(t)$

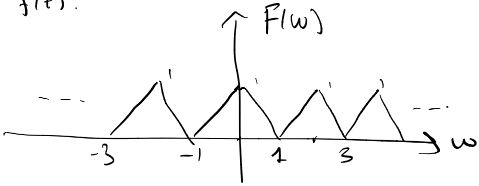
$Y(\omega) = \frac{1}{2\pi} Z(\omega) * \pi [\delta(\omega-1) + \delta(\omega+1)]$

$= \frac{1}{2} Z(\omega-1) + \frac{1}{2} Z(\omega+1)$

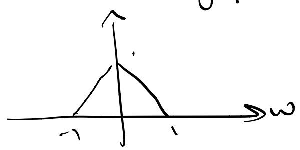




5) Suppose $F(\omega)$ is as shown below. What is its inverse Fourier transform $f(t)$?



A Let $G(\omega)$ be the graph below



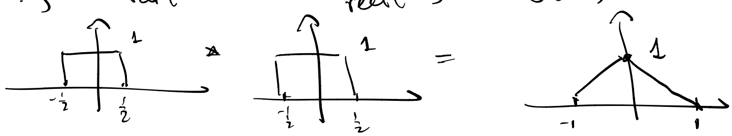
Then $F(\omega) = \sum_{n=-\infty}^{\infty} G(\omega - 2n)$

$= G(\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - 2n)$

So $f(t) = 2\pi\, \mathcal{F}^{-1}\{G(\omega)\} \cdot \mathcal{F}^{-1}\{\sum_{n=-\infty}^{\infty} \delta(\omega-2n)\}$

First! $\mathcal{F}^{-1}\{G(\omega)\}$ $\quad$ $\text{rect}(\omega) * \text{rect}(\omega) = G(\omega)$

We know



So $\mathcal{F}^{-1}\{G(\omega)\} = 2\pi \cdot \mathcal{F}^{-1}\{\text{rect}(\omega)\} \cdot \mathcal{F}^{-1}\{\text{rect}(\omega)\}$

$\sin(at) \longleftrightarrow 2\pi\ \text{rect}(\omega/2a)$ $\qquad 2a = 1$

$\frac{\text{sinc}(t/2)}{2\pi} \longleftrightarrow \text{rect}(\omega)$ $\qquad a = 1/2$

Then $\mathcal{F}^{-1}\{G(\omega)\} = 2\pi \left[\frac{1}{2\pi}\text{sinc}(t/2)\right]^2$

$= \frac{1}{2\pi}\text{sinc}^2(t/2)$

Also $\frac{1}{2}\sum_{n=-\infty}^{\infty} \delta(t - \pi n) \longleftrightarrow \sum_{n=-\infty}^{\infty} \delta(\omega - 2n)$

Then $f(t) = \frac{1}{2\pi}\text{sinc}^2(t/2) \cdot \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta(t-\pi n) \cdot 2\pi$

$= \boxed{\frac{1}{2}\sum_{n=-\infty}^{\infty} \text{sinc}^2\left(\frac{\pi}{2}n\right)\delta(t-\pi n)}$

(6) If $x(t) = e^{-at}u(t) + e^{at}u(-t)$, what is $\mathcal{L}\{x(t)\}$?
(Suppose $Re(a) > 0$)

A $\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$

$= \int_{-\infty}^{\infty}[e^{-at}u(t) + e^{at}u(-t)]\,e^{-st}\,dt$

$= \int_{0}^{\infty} e^{-at}e^{-st}\,dt + \int_{-\infty}^{0} e^{at}e^{-st}\,dt$

$= \int_{0}^{\infty} e^{-t(s+a)}\,dt + \int_{-\infty}^{0} e^{-t(s-a)}\,dt$

$= -\frac{1}{s+a}\left[e^{-\infty(s+a)} - e^{0}\right] - \frac{1}{s-a}\left[e^{0} - e^{\infty(s-a)}\right]$

$= -\frac{1}{s+a}[-1] - \frac{1}{s-a}[1]$

if $Re(s+a) > 0$ and $Re(s-a) < 0$

$$\begin{cases} Re(s) > -Re(a) \\ R(s) < Re(a) \end{cases} \Rightarrow \boxed{-Re(a) < Re(s) < Re(a) \quad ROC}$$

$= \frac{1}{s+a} - \frac{1}{s-a}$

$= \frac{s-a-s-a}{s^2-a^2}$

$= -\frac{2a}{s^2-a^2}$

$= \boxed{\frac{2a}{a^2-s^2}}$ with R.O.C. $-Re(a) < Re(s) < Re(a)$