Using the Fourier Transform to Find $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n}$

Fnu Kalkin
fkalkin@ucsd.edu

June 20, 2023

We will derive the value of this sum for all $x \in \mathbb{R}_{\geq 0}$ using fourier transforms.

Proof. First consider the convolution of an impulse train and a carefully chosen sinc wave, where k is some positive real number.

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{\infty} \delta(t - 2\pi n) * \frac{k}{\pi}\text{sinc}\left(\frac{k}{\pi}t\right) \\ &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{k}{\pi}(t - 2\pi n)\right) \\ &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{k}{\pi}t - 2kn\right) \qquad (1) \end{aligned}$$

Note that a convolution in the time domain is a product in the frequency domain. So,

$$\begin{aligned} Y(\omega) &= \mathcal{F}\left(\sum_{n=-\infty}^{\infty} \delta(t - 2\pi n)\right) \cdot \mathcal{F}\left(\frac{k}{\pi}\text{sinc}\left(\frac{k}{\pi}t\right)\right) \\ &= \sum_{n-\infty}^{\infty} \delta(w - n) \cdot \frac{k}{\pi}\frac{\pi^2}{k}\text{rect}\left(\frac{\pi\omega}{2k}\right) \\ &= \pi \sum_{n-\infty}^{\infty} \delta(w - n) \cdot \text{rect}\left(\frac{\pi\omega}{2k}\right) \end{aligned}$$

Note that for all n such that $|n| > \frac{k}{\pi}$, the rect function effectively "zeroes out" the delta centered at $w = n$. Since these deltas are centered at discrete values of n, $|n|$ can be at most $\left\lfloor \frac{k}{\pi} \right\rfloor$. Thus,

$$Y(\omega) = \pi \sum_{n=-\lfloor k/\pi \rfloor}^{\lfloor k/\pi \rfloor} \delta(w - n)$$

For $n \neq 0$, this is just the fourier transform of a sum of cosine waves. We take the inverse fourier transform of $Y(\omega)$ to find $y(t)$:

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left(Y(\omega)\right) \\ &= \mathcal{F}^{-1}\left(\pi\delta(\omega)\right) + \sum_{n=1}^{\lfloor k/\pi \rfloor} \cos(nt) \\ &= \frac{1}{2} + \sum_{n=1}^{\lfloor k/\pi \rfloor} \cos(nt) \end{aligned}$$

And this must be equal to the earlier result (1), when we simply took the convolution instead. So,

$$\frac{1}{2} + \sum_{n=1}^{\lfloor k/\pi \rfloor} \cos(nt) = \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left(\frac{k}{\pi}t - 2kn\right) \tag{2}$$

Which is an intriguing result by itself.

To get the desired summation, simply plug in $t = 0$ into equation (2) and use the fact that sinc is an even function.

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\lfloor k/\pi \rfloor} 1 &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(-2kn) \\
\frac{1}{2} + \lfloor k/\pi \rfloor &= \frac{k}{\pi} \sum_{n=-\infty}^{\infty} \operatorname{sinc}(2kn) \\
\frac{1}{2} + \lfloor k/\pi \rfloor &= \frac{k}{\pi} \left(\operatorname{sinc}(0) + \sum_{n=1}^{\infty} \operatorname{sinc}(2kn) + \operatorname{sinc}(-2kn) \right) \\
\frac{\pi}{k} \left(\frac{1}{2} + \lfloor k/\pi \rfloor \right) &= 1 + 2 \sum_{n=1}^{\infty} \operatorname{sinc}(2kn) \\
2 \sum_{n=1}^{\infty} \frac{\sin(2kn)}{2kn} &= \frac{\pi}{k} \left(\frac{1}{2} + \lfloor k/\pi \rfloor \right) - 1 \\
\sum_{n=1}^{\infty} \frac{sin(2kn)}{n} &= \pi \left(\frac{1}{2} + \lfloor k/\pi \rfloor \right) - k
\end{aligned}$$

Since k is just some positive real number, we can let $x = 2k$. Then,

$$\sum_{n=1}^{\infty} \frac{\sin(xn)}{n} = \pi \left(\frac{1}{2} + \left\lfloor \frac{x}{2\pi} \right\rfloor \right) - \frac{x}{2}$$

$\square$