Using the Poisson Summation Formula to Find $\sum_{n=1}^{\infty} \frac{\sin(xn)}{n}$

Kalkin Kalkin
fkalkin@ucsd.edu

May 31, 2025

First, let's define the notation to be used in this document. If $f \in \mathrm{L}^1(\mathbb{R})$, we define the *Fourier transform* $\widehat{f}$ of f by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i \xi t} dt$$

Let $f \in \mathrm{L}^1(\mathbb{R})$. We define $Pf(x) = \sum_{k \in \mathbb{Z}} f(x+k)$. Then $Pf \in \mathrm{L}^1(\mathbb{T})$ and $\widehat{Pf}(k) = \widehat{f}(k)$ for all $k \in \mathbb{Z}$ by Theorem 8.31 in Folland's *Real Analysis*.
Now we state and prove a special case of the Poisson Summation formula. The conditions stated in Folland's *Real Analysis* don't quite hold for the function I will define later in this document so I will prove a weak version that works for our purposes here.

Theorem 1. Let $f \in \mathrm{L}^1(\mathbb{R})$ such that Pf is continuous at 0 and $Pf \in \mathrm{BV}(\mathbb{T})$. Then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)$$

Proof. By Theorem 8.43 in Folland, $Pf(0) = \sum_{n=\infty}^{\infty} \widehat{Pf}(n)$. Plugging in definitions, we get

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)$$

. □

Now, we proceed with calculating the desired sum.

Theorem 2. If $x \in \mathbb{R}$ such that $x \neq 2\pi k$ for some nonzero integer k, then

$$\sum_{n=1}^{\infty} \frac{\sin(xn)}{n} = \pi \left(\frac{1}{2} + \left\lfloor \frac{x}{2\pi} \right\rfloor \right) - \frac{x}{2}$$

Proof. Let $f_x(t) = \chi_{[-\frac{x}{2\pi}, \frac{x}{2\pi}]}$, the characteristic function of $[-\frac{x}{2\pi}, \frac{x}{2\pi}]$.
Clearly $f_x(t) \in \mathrm{L}^1$, and so

$$\begin{aligned}
\widehat{f_x}(\xi) &= \int_{-\infty}^{\infty} f_x(t) e^{-2\pi i \xi t} dt \\
&= \int_{-\frac{x}{2\pi}}^{\frac{x}{2\pi}} e^{-2\pi \xi t} dt \\
&= -\frac{1}{2\pi i \xi} \left(e^{-x\xi} - e^{x\xi} \right) \\
&= \frac{1}{2\pi i \xi} \left(2i \sin(x\xi) \right) \\
&= \frac{\sin(x\xi)}{\pi \xi}
\end{aligned}$$

by a routine application of Euler's formula.
It's obvious that the conditions for Theorem 1 hold: Pf can only take on finitely many values, so it is of bounded variation, and continuity at 0 holds whenever x is not a nonzero multiple of 2π. So by Theorem 1,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f_x(n) &= \sum_{n=-\infty}^{\infty} \widehat{f_x}(n) \\
\sum_{n=-\infty}^{\infty} \chi_{[-\frac{x}{2\pi}, \frac{x}{2\pi}]}(n) &= \sum_{n=-\infty}^{\infty} \frac{\sin(xn)}{\pi n}
\end{aligned}$$

whenever $x \neq 2\pi k$, $k \in \mathbb{Z} \setminus \{0\}$. On the left, we have $f_x(n) = 1$ whenever $|n| \leq \frac{x}{2\pi}$, and zero otherwise. So $f_x(n)$ is nonzero for precisely $1 + 2 \cdot \lfloor \frac{x}{2\pi} \rfloor$ different values of n. Then,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{\sin(xn)}{\pi n} &= 1 + 2 \left\lfloor \frac{x}{2\pi} \right\rfloor \\
x + 2 \sum_{n=1}^{\infty} \frac{\sin(xn)}{n} &= \pi \left(1 + 2 \left\lfloor \frac{x}{2\pi} \right\rfloor \right) \\
\sum_{n=1}^{\infty} \frac{\sin(xn)}{n} &= \pi \left(\frac{1}{2} + \left\lfloor \frac{x}{2\pi} \right\rfloor \right) - \frac{x}{2}
\end{aligned}$$

So we are done. $\square$